GIRRAWEEN HIGH SCHOOL

HALF YEARLY EXAMINATIONS

2017

MATHEMATICS

EXTENSION 1

Time Allowed: Two hours

(Plus 5 minutes reading time)

Instructions:

- There are 10 questions in this paper. All questions are compulsory.
- Use blue or black pen.
- Write all your answers in the Answer Booklets provided.
- For Questions 1 – 5, fill in the circle corresponding to the correct answer in your answer booklet.
- For Questions 6 – 10, start each question on a new page.
- Write on both sides of the paper.
- Show all necessary working.
- Board-approved calculators may be used.
- Mathematics reference sheets are provided.
- Marks may be deducted for careless or badly arranged work.
- Write 'End of Solutions' at the conclusion of your solutions to the task.

Multiple Choice

For Questions 1 – 5, fill in the circle corresponding to the correct answer in your answer booklet.

1. What is the size of the angle between the lines $2x - y = 0$ and $x + y = 0$ correct to the nearest degree?

   A. 18° B. 19° C. 71° D. 72°

2. $A(-2, 5)$ and $B(4, -1)$ are two points on the number plane. What are the coordinates of $P(x, y)$ that divides *AB* internally in the ratio 2:1?

   A. $(-5, 8)$ B. $(0, 3)$ C. $(2, 1)$ D. $(7, -4)$

3. Which of the following is an expression for $\cos(A - B) - \cos(A + B)$?

   A. $2\sin A\sin B$ B. $2\cos A\cos B$
   C. $-\cos A\cos B$ D. $-2\sin A\sin B$

4. In how many ways can 10 boys be arranged in a straight line if the first boy in the line is Martin and the last boy is Edward?

   A. 80 640 B. 40 320

   C. 3 628 800 D. 7 257 600

5. The equation $2x^3 + x^2 - 13x + 6 = 0$ has roots α, $\frac{1}{\alpha}$ and β.
   What is the value of β?

   A. 3 B. 2 C. -3 D. -6

Question 6 (17 marks)

a. Solve the inequality $\frac{x+4}{x-3} \leq 2$. [3]

b. The interval *AB* has endpoints *A* (3, 2) and *B* (4, 5). Find the coordinates of the point *P* which divides the interval *AB* externally in the ratio 3: 4. [3]

c. Find the value of *m*, where *m* > 0, such that the acute angle between the lines $y = 2x$ and $y = mx$ is 45°. [3]

d. (i) Express $\cos x - \sqrt{3}\sin x$ in the form $A\cos(x + \alpha)$ where $A > 0$ and α is an acute angle. [3]

(ii) Hence or otherwise, solve the equation [2]

$$\cos x - \sqrt{3}\sin x = 2 \text{ for } 0° \leq x \leq 360°.$$

e. Show that the expression $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$ is independent of θ. [3]

(where $\sin\theta \neq 0$ and $\cos\theta \neq 0$)

Question 7 (13 marks)

a. Consider the function $f(x) = \frac{e^x}{3 + e^x}$.

Note that e^x is always positive and that $f(x)$ is defined for all real x.

(i) Show that $f(x)$ has no stationary points. [2]

(ii) Show that $f''(x) = \frac{3e^x(3 - e^x)}{(3 + e^x)^3}$ [3]

(ii) Find the coordinates of the point of inflexion. [3]

b. Differentiate $\log_2 x^2$ [2]

c. Find the volume generated when $y = \log_e x$ is rotated about the y-axis between $y = 1$ and $y = 3$. Express your answer in exact form. [3]

Question 8 (16 marks)

a. When the polynomial $P(x) = (x - 1)(x - 2)\,Q(x) + 3x + k$ is divided by

$(x - 1)$, the remainder is -1. Find the remainder when $P(x)$ is divided by $(x - 2)$ [3]

b. The equation $\tan^2\theta + b\tan\theta + c = 0$ has roots $\tan\alpha$ and $\tan\beta$.

Find the expression for $\tan(\alpha + \beta)$ in terms of b and c. [2]

c. Find the term independent of x in the expansion of $\left(3x^2 + \frac{2}{x}\right)^6$. [3]

d. Five different fair dice are thrown together. Find the probability that

(i) the five scores are all different [2]

(ii) the five scores include at most one 6. [3]

e. In how many ways can 11 people occupy seats at two circular tables, where one table

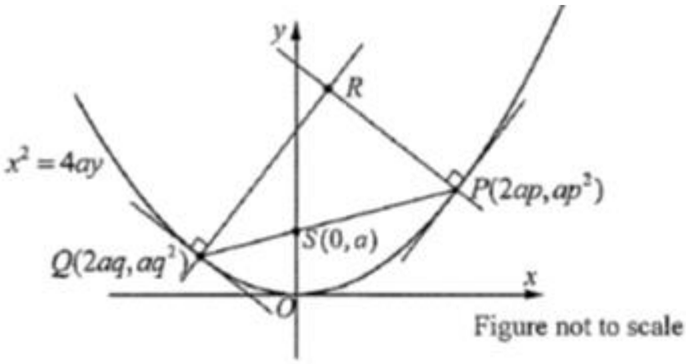
can seat 6 people and the other can seat 5 people. [3]

Question 9 (13 marks)

a. The points $P(2ap,\ ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$

such that PQ is a focal chord.

The normal at P and Q intersect at R.



(i) Show that the equation of the normal at P is given by $x + py = ap^3 + 2ap$. [3]

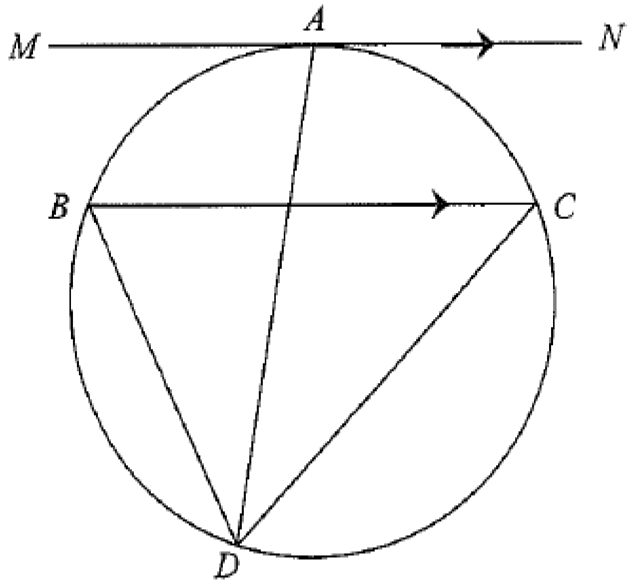
(ii) Show that R is the point $(-apq(p + q), a(p^2 + q^2 + pq + 2))$. [3]

(iii) Hence, show that the equation of the locus of R as P and Q move on

the parabola is given by $x^2 = a(y - 3a)$ [3]

Question 9 continues on the next page…

b.



In the diagram, *MAN* is tangent to the circle at *A*. *BC* is a chord of the circle such that *BC* //*MN.*

D is a point on the circle.

(i) Copy the diagram into your answer booklet.

(ii) Show that AD bisects $\angle BDC$ [4]

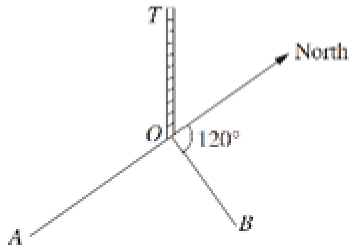
Examination continues on the next page...

Question 10 (9 marks)

a. Use the principal of Mathematical Induction to prove that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!} \quad \text{for } n \geq 1$$ [4]

b.



From a point A due south of a tower, the angle of elevation of the top of the tower T, is 23°. From another point B, on a bearing of 120° from the tower, the angle of elevation of T is 32°. The distance AB is 200 metres.

(i) Copy the diagram into your writing booklet and add the given information. [1]

(ii) Find the height of the tower. [4]

End of Examination

MC

1. D 2. C 3. A 4. B 5. C

Q1

$\tan\theta = \left|\frac{2-(-1)}{1+2(-1)}\right| = 3$ **D**

Q2. A(−2, 5) B(4, −1)

2 : 1

$x = \frac{2(4)+1(-2)}{3}$; $y = \frac{2(-1)+1(5)}{3}$ **C**

Q3 $\cos(A-B)-\cos(A+B)$

$= \cos A\cos B + \sin A\sin B - [\cos A\cos B - \sin A\sin B]$

A

Q4 8! **B**

Q5 Product of roots : $\alpha \cdot \frac{1}{\alpha} \cdot \beta = \beta = -\frac{6}{2}$

C

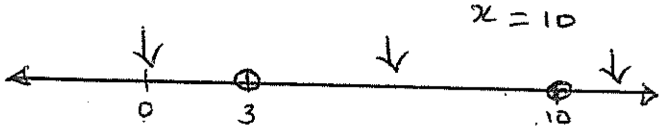
Question 6 (17 marks)

a. $\frac{x+4}{x-3} \le 2$

$x \ne 3$; Solve $\frac{x+4}{x-3} = 2$

$2x - 6 = x + 4$

$x = 10$



Test $x=0$: $\frac{4}{-3} \le 2$ ✓ Test $x=5$: $\frac{9}{2} \le 2$ ✗ Test $x=12$: $\frac{16}{9} \le 2$ ✓

∴ Solution : $x < 3$, $x \ge 10$ **3m**

b. A(3, 2) B(4, 5)

−3 : 4

$x = \frac{-3(4)+4(3)}{-3+4}$; $y = \frac{-3(5)+4(2)}{-3+4}$

P(0, −7) **3m**

c. $y = 2x$; $y = mx$

$m_1 = 2$; $m_2 = m$

$\tan 45° = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

$1 = \left|\frac{2-m}{1+2m}\right|$

$1+2m = 2-m$ or $1+2m = -(2-m)$

$3m = 1$ $1+2m = -2+m$

$m = \frac{1}{3}$ $m = -3$

∴ $m = \frac{1}{3}$ (since $m > 0$) **3m**

d. (i) $\cos x - \sqrt{3}\sin x = A\cos(x+\alpha)$

$= A\cos x\cos\alpha - A\sin x\sin\alpha$

Equating coefficients,

$A\cos\alpha = 1$ —① & $A\sin\alpha = \sqrt{3}$ —②

Squaring and adding ① & ②

$A^2(\cos^2\alpha + \sin^2\alpha) = 4$

$A^2 = 4$, $A = 2$ $(A > 0)$

∴ $2\cos\alpha = 1$; $2\sin\alpha = \sqrt{3}$ ∴ $\alpha < 90°$

$\alpha = \frac{\pi}{3}$

∴ $\cos x - \sqrt{3}\sin x = 2\cos\left(x+\frac{\pi}{3}\right)$ **3m**

ii) $\cos x - \sqrt{3}\sin x = 2\cos\left(x+\frac{\pi}{3}\right) = 2$

$\cos\left(x+\frac{\pi}{3}\right) = 1$

$x + \frac{\pi}{3} = 0, 2\pi, 4\pi$

$x = \frac{5\pi}{3}$ **2m**

e) $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$

$= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$

$= \frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta} = \frac{\sin 2\theta}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$

$= 2$ which is independent of θ

a. $f(x) = \dfrac{e^x}{3+e^x}$

i) $f'(x) = \dfrac{(3+e^x)e^x - e^x \cdot e^x}{(3+e^x)^2}$

$= \dfrac{3e^x + e^{2x} - e^{2x}}{(3+e^x)^2}$

$= \dfrac{3e^x}{(3+e^x)^2}$

> 0 for all values of x since $e^x > 0$

$\therefore f'(x)$ has no stationary points [2m]

ii) $f''(x) = \dfrac{3e^x(3+e^x)^2 - 3e^x \cdot 2(3+e^x) \cdot e^x}{(3+e^x)^4}$

$= \dfrac{3e^x(3+e^x)\left[3+e^x - 2e^x\right]}{(3+e^x)^4}$

$= \dfrac{3e^x(3-e^x)}{(3+e^x)^3}$ [GIVEN]

Point of inflexion when $f''(x) = 0$

ie. $\dfrac{3e^x(3-e^x)}{(3+e^x)^3} = 0$

$\therefore 3 - e^x = 0$

$e^x = 3$

$x = \ln 3$

At $x = \ln 3$, $y = \frac{3}{3+3} = \frac{1}{2}$

Possible point of inflexion at $(\ln 3, \frac{1}{2})$

x	$\ln 2$	$\ln 3$	$\ln 4$
$f''(x)$	$\frac{6}{5^3} > 0$	0	$\frac{-7}{7^3} < 0$

There is a change in concavity $\therefore (\ln 3, \frac{1}{2})$ is a point of inflexion [3m]

$f(x) = \dfrac{e^x}{3+e^x} < \dfrac{3+e^x}{3+e^x} = 1$

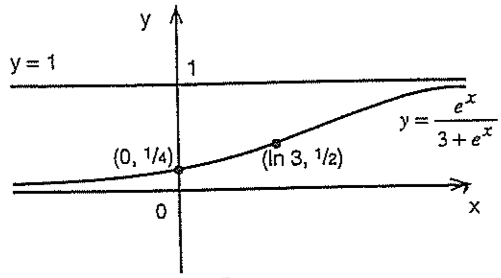
$\therefore f(x) < 1$

$\therefore 0 < f(x) < 1$ [2m]

iv) As $x \to \infty$, $e^x \to \infty$, $f(x) \to 1$

As $x \to -\infty$, $e^x \to 0$, $f(x) \to 0$ [2m]

v)



b. $y = \log_2 x^2 = \dfrac{\log_e x^2}{\log_e 2}$

$= \dfrac{1}{\log_e 2} \cdot 2\log_e x$

$\dfrac{dy}{dx} = \dfrac{1}{\log_e 2}\left[\dfrac{d}{dx} 2\log_e x\right]$

$= \dfrac{1}{\log_e 2} \cdot \dfrac{2}{x}$

$= \dfrac{2}{x \log_e 2}$ [2m]

c. $y = \log_e x \Rightarrow x = e^y \Rightarrow x^2 = e^{2y}$

$V = \pi \int_1^3 x^2 \, dy$

$= \pi \int_1^3 e^{2y} \, dy$

$= \dfrac{\pi}{2}\left[e^{2y}\right]_1^3$

$= \dfrac{\pi}{6}\left[e^6 - e^2\right]$ cubic units [3m]

Question 8 (13 marks)

a. $P(x) = (x-1)(x-2).Q(x) + 3x + k$

$P(1) = 3(1) + k = -1$

$\therefore k = -4$

$\therefore P(x) = (x-1)(x-2).Q(x) + 3x - 4$

$P(2) = 3(2) - 4$

$= 2$

Remainder $= 2$ — 2m

b. $\tan^2\theta + b\tan\theta + c = 0$

sum of roots : $\tan\alpha + \tan\beta = -b$

product of roots : $\tan\alpha\tan\beta = c$

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{-b}{1-c}$ — 2m

c. $\left(3x^2 + \frac{2}{x}\right)^6$

$T_{k+1} = {}^6C_k (3x^2)^{6-k}\left(\frac{2}{x}\right)^k$

$= {}^6C_k\, 3^{6-k} x^{12-2k} . 2^k . x^{-k}$

$= {}^6C_k\, 3^{6-k}\, 2^k\, x^{12-3k}$

For term independent of x,

$12 - 3k = 0$

$k = 4$

$\therefore T_5 = {}^6C_4\, 3^2 2^4 = 2160$

is the term independent of x. — 3m

d(i) Total : 6^5

All 5 different : $6\times5\times4\times3\times2$

P(all different) $= \frac{5}{54}$ — 2m

ii) At most one 6 ⇒ 1 six or 0 sixes

P(at most 1 six) $= {}^5C_0\left(\frac16\right)^0\left(\frac56\right)^5 + {}^5C_1\left(\frac16\right)^1\left(\frac56\right)^4$

$= \left(\frac56\right)^5 + 5\left(\frac{5^4}{6^5}\right)$

$= 2\left(\frac56\right)^5$ — 3m

e. ${}^{11}C_6 \times 5! \times 4!$ — 3m

select 6 people; arrange around table 1; arrange around table 2.

Question 9 (13 marks)

a. i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At $P(2ap, ap^2)$, $m_{tangent} = \frac{2ap}{2a}$

$= p$

$\therefore m_{normal} = -\frac{1}{p}$

E_{normal} : $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$ — ①

3m

ii) Similarly, E_{normal} at Q

$x + qy = aq^3 + 2aq$ — ②

① − ②

$(p-q)y = a(p^3 - q^3) + 2a(p-q)$

$= a(p-q)(p^2+pq+q^2) + 2a(p-q)$

$y = a(p^2+pq+q^2) + 2a$

$= a(p^2+pq+q^2+2)$ — ③

$$x + pa(p^2+q^2+pq+2) = ap^3 + 2ap$$

$$x + ap^3 + apq^2 + ap^2q + 2ap = ap^3 + 2ap$$

$$x = -ap^2q - apq^2$$

$$= -apq(p+q)$$

$$\therefore R\left(-apq(p+q),\ a(p^2+pq+q^2+2)\right)$$

[3m]

iii) PQ is a focal chord $\therefore pq = -1$

At R,

$x = -apq(p+q)$; $y = a(p^2+pq+q^2+2)$

$x = a(p+q)$; $y = a(p^2+q^2+1)$

$\therefore p+q = \frac{x}{a}$; $y = a\left((p+q)^2 - 2pq + 1\right)$

$$= a\left(\left(\frac{x}{a}\right)^2 + 2 + 1\right)$$

$$= a\left(\frac{x^2}{a^2} + 3\right)$$

$$y = \frac{x^2 + 3a^2}{a}$$

$$ay = x^2 + 3a^2$$

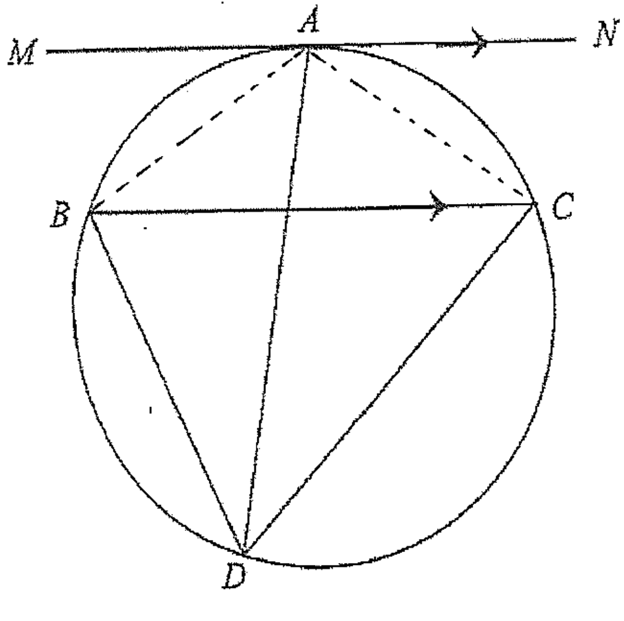
$$x^2 = ay - 3a^2$$

$$\therefore x^2 = a(y - 3a)$$

[3m]

b.

i)



$\angle CDA = \angle CBA$ ($\angle$s subtended by arc AC at the circumference)

$\angle CBA = \angle MAB$ (alternate $\angle$s, BC $\parallel$ MN)

$\angle MAB = \angle ADB$ ($\angle$ in the alternate segment)

$\therefore \angle CDA = \angle ADB$

$\therefore$ AD bisects $\angle BDC$.

[4m]

Question 10 (9 marks)

a. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)}$ for $n \geq 1$

Step 1: Show true for $n = 1$

LHS $= \frac{1}{2!} = \frac{1}{2}$; RHS $= 1 - \frac{1}{(1+1)!}$

$= \frac{1}{2}$

LHS = RHS

$\therefore$ true for $n = 1$

Step 2: Assume true for $n = k$

ie. $\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$

Step 3: Prove true for $n = k+1$

ie.

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{(k+1)}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

$$\text{LHS} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \quad \text{(from step 2)}$$

$$= 1 - \frac{k+2}{(k+2)(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= 1 - \frac{k+2-(k+1)}{(k+2)!}$$

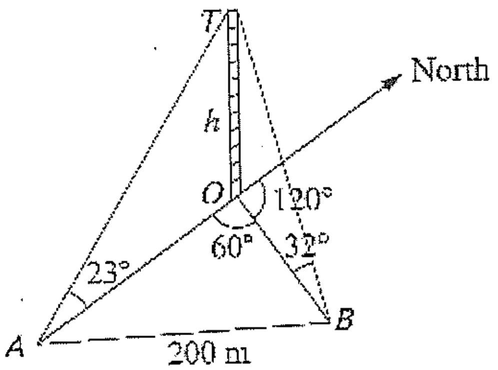
$$= 1 - \frac{1}{(k+2)!} = \text{RHS}.$$

also true for $n = k+1$.

Step 4: By the principle of Mathematical Induction, the result is true for $n \geq 1$.

4m

b. i)



1m

ii) let height of tower $OT = h$

In ΔOAT, $\tan 23° = \frac{h}{OA}$

$\therefore OA = \frac{h}{\tan 23°}$

In ΔOBT, $\tan 32° = \frac{h}{OB}$

$\therefore OB = \frac{h}{\tan 32°}$

In ΔAOB, using cosine rule

$$200^2 = OA^2 + OB^2 - 2 \times OA \times OB \cos 60°$$

$$= \frac{h^2}{\tan^2 23°} + \frac{h^2}{\tan^2 32°} - \frac{2h^2}{\tan 23° \tan 32°} \cdot \frac{1}{2}$$

$$= h^2\left(\frac{1}{\tan^2 23°} + \frac{1}{\tan^2 32°} - \frac{1}{\tan 23° \tan 32°}\right)$$

$$= h^2 (4.340\ldots)$$

$$h^2 = \frac{200^2}{4.340\ldots}$$

$$h = 95.99\ldots$$

$h = 96$ m (to the nearest m)

4m